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Zeros of the Jones Polynomial for Multiple Crossing-Twisted Links

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Abstract Let D be a general connected reduced alternating link diagram, C be the set of crossings of D and C' be the nonempty subset of C. In this paper we first define a multiple crossing-twisted link family $\{D^n(C')|n = 1, 2, \ldots\}$ based on D and C', which produces $(2, 2n + 1)$-torus knot family, the link family A_n defined in Chang and Shrock (Physica A 301:196–218, 2001) and the pretzel link family $P(n, n, n)$ as special cases. Then by applying Beraha-Kahane-Weiss's Theorem we prove that limits of zeros of Jones polynomials of $\{D^n(C')|n = 1, 2, \ldots\}$ are the unit circle $|z| = 1$ (It is independent of the selections of D and C') and several isolated limits, which can be determined by computing flow polynomials of subgraphs of G corresponding to D. Furthermore, we use the method of Brown and Hickman (Discrete Math. 242:17–30, 2002) to prove that, for any $\epsilon > 0$, all zeros of Jones polynomial of the link $D^n(C)$ lie inside the circle $|z| = 1 + \epsilon$, provided that n is large enough. Our results extend results of F.Y. Wu, J. Wang, S.-C. Chang, R. Shrock and the present authors and refine partial result of A. Champanerkar and L. Kofman.

Keywords Jones polynomial · Zeros · Twisted links · Unit circle theorem · Potts model

1 Introduction

Zeros of the Jones polynomial are interesting since it is the special case of partition functions of the Potts model in physics [11, 18]. The study of zeroes in physics originated from two very well-known papers [14, 20] on phase transitions by T.D. Lee and C.N. Yang. One hope to gain much information by considering complex variables and studying zeros. In [19] and [5], F.Y. Wu, J. Wang, S.-C. Chang and R. Shrock initiated the study of zeros of the Jones polynomial. In this paper we shall study zeros of the Jones polynomial for multiple crossing-twisted links. Now we give the definition.

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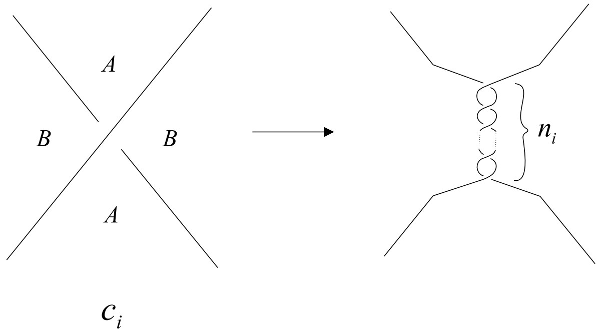


Fig. 1 Twist crossing c_i round the role through the A channels n_i times

Given an unoriented link diagram, every crossing of the diagram distinguishes two out of the four small regions incident at the crossing. Rotating the over-crossing arc counterclockwise until the under-crossing arc is reached, and call the two small regions swept out the A channels and the other two the B channels.

Let D be a connected alternating link diagram, $C = \{c_1, c_2, \dots, c_m\}$ be the set of crossings of D. Twist each crossing c_i round the role through the two A channels of the crossing n_i times as shown in Fig. 1, we obtain an alternating link diagram $D(n_1, n_2, \dots, n_m)$ and call it a multiple crossing-twisted link diagram. Let $C' \subset C$. We denote by $D^n(C')$ the multiple crossing-twisted link diagram obtained from D by twisting each crossing in C' n times. Trivially, $D^1(C) = D$. Thus, given an alternating link diagram D and a subset C' of its crossings, we define a multiple crossing-twisted link family $\{D^n(C')|n = 1, 2, \dots\}$.

For example, if we take D to be the diagram of the left trefoil knot and $C' = \{c_3\}$, we obtain knots family A_n in [5]; if we take D to be the diagram of the right trefoil knot and $C' = \{c_1, c_3\}$, we obtain the $(2, 2n + 1)$-torus knots family [19] and if we take D to be the diagram of the left trefoil knot and $C' = \{c_1, c_2, c_3\}$, we obtain the pretzel link family $P(n, n, n)$ [6]. See Fig. 2(a), (b) and (c), respectively. In the above three references, the authors showed that zeros of Jones polynomials of these three link families are all dense in the unit circle $|z| = 1$. It is natural to ask if this phenomenon happens in general or can we get the same result for any connected reduced alternating link diagram D. Now we show that the answer is "yes".

In this paper by applying Beraha-Kahane-Weiss's Theorem [1] we shall prove that for any D and C', limits of zeros of Jones polynomials of the multiple crossing-twisted link family $\{D^n(C')|n = 1, 2, \dots\}$ are the unit circle $|z| = 1$ (It is independent of the selections of D and C') and several isolated limits, which can be determined by computing flow polynomials of subgraphs of G constructed from D. Furthermore, we use the method of [3] to prove that, for any $\epsilon > 0$, there is a natural number N_ϵ such that if $n > N_\epsilon$, all zeros of Jones polynomial of the link $D^n(C)$ lie inside the circle $|z| = 1 + \epsilon$. Our results extend results of F.Y. Wu, J. Wang [19], S.-C. Chang, R. Shrock [5] and the present authors [6], and refine partial result of A. Champanerkar and L. Kofman [4].

Originally, the Jones polynomial was given in terms of the trace of a matrix representation of the braid group into a Temperley-Lieb algebra [9]. In 1987, L.H. Kauffman gave a state model for the Jones polynomial using his bracket polynomial [12], which provided another way of calculating the Jones polynomial. Let $V_L(t)$ be the Jones polynomial of an oriented link L. Let D be a link diagram of the oriented link L, and $[D]$ be the Kauffman bracket polynomial in one variable A of D with orientations neglected. Then

$$V_L(t) = (-A^3)^{-w(D)}[D]|_{A=t^{-1/4}}, \tag{1}$$

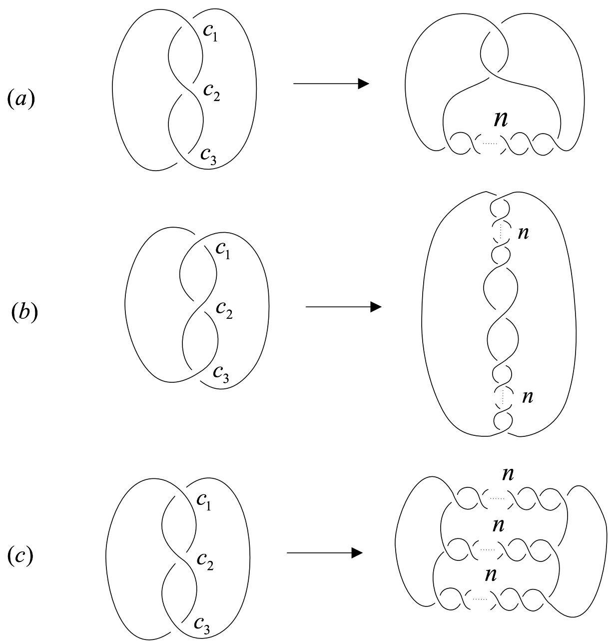


Fig. 2 Constructions of the link family (**a**) A_n, (**b**) $(2, 2n+1)$-tours knot, and (**c**) $P(n, n, n)$

where $w(D)$ is the writhe of D.

The Jones polynomial is not a polynomial in the common sense since the degrees of its terms may be negative and half-integers. Actually, when the number of components of L is odd, the degree of each term of $V_L(t)$ is an integer; and when the number of components of L is even, the degree of each term of $V_L(t)$ is a half-integer [10]. For each case, we can always turn Jones polynomial into a polynomial by extracting the term with the lowest degree. We shall consider the Jones polynomial in this sense, which implies that $V_L(0) \neq 0$ for any link L. Throughout the paper we use the notation $\doteq$ to denote equality up to a factor $\pm t^k$, where k is an integer or a half integer.

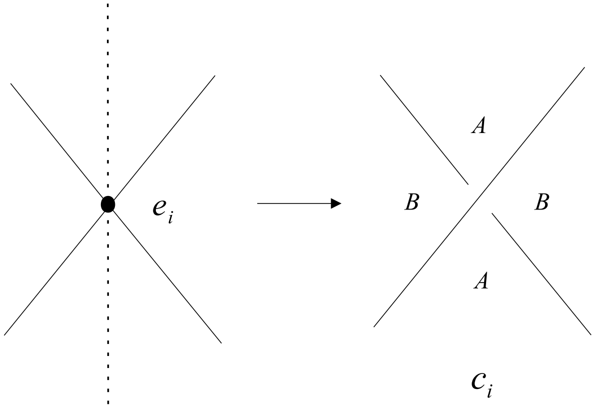
2 Graph Theory Background

Let D be a connected alternating link diagram. Then each of its regions has only A-channels or only B-channels. Calling a region an A-region if all its channels are A-channels, and a B-region if all its channels are B-channels. Now construct a connected plane graph G from D as follows. For each A-region R, take a vertex V_R in R, and for each crossing at which R_1 and R_2 meet, take an edge $V_{R_1}V_{R_2}$. It's clear that G is a connected plane graph.

Conversely, give a connected plane graph G, we first construct the medial graph $M(G)$ of G as follows. Insert a vertex on every edge of G, and join two new vertices by an edge lying in a face of G if the vertices are on adjacent edges of the face. Then turn each vertex of $M(G)$ to a crossing as shown in Fig. 3. We obtain a connected alternating diagram D.

Thus, we build a one-to-one correspondence between connected link diagrams and connected plane graphs. It is clear that crossings of D correspond to edges of G. Now suppose that $E = \{e_1, e_2, \ldots, e_m\}$ is the edge set of the graph G with e_i corresponding to the crossing c_i for $i = 1, 2, \ldots, m$. If $G(n_1, n_2, \ldots, n_m)$ is the connected plane graph corresponding to

Fig. 3 Turn the vertex of $M(G)$ on the edge e_i (*dashed line*) of G to the crossing c_i of D



the diagram $D(n_1, n_2, \ldots, n_m)$, then $G(n_1, n_2, \ldots, n_m)$ can be obtained from G by subdividing the edge e_i $n_i - 1$ times for $i = 1, 2, \ldots, m$. Let $E' \subset E$ be the edge subset of G corresponding to C' of D. If $G^n(E')$ is the connected plane graph corresponding to $D^n(C')$, then $G^n(E')$ can be obtained from G by subdividing each edge in E' $n - 1$ times.

In this paper, we shall always assume that D is a connected reduced alternating link diagram. Note that the nugatory crossings of diagrams correspond to the bridges and loops of plane graphs. Accordingly, G is a 2-edge connected loopless plane graph.

Let $G = (V, E)$ be a graph. We denote by $\mathrm{comp}(G)$ the number of connected components of the graph G, and by $c(G)$ the nullity of the graph G, i.e. $c(G) = |E| - |V| + \mathrm{comp}(G)$. We shall also use some basic properties on the flow polynomial in this paper and refer the readers to see [17] or Sect. 3 of [16]. We follow [2] for undefined terminology and notations of graphs.

3 The Jones Polynomial of $D^n(C')$

In [13], L.H. Kauffman converted the Kauffman bracket polynomial of link diagrams to the Tutte polynomial of corresponding signed plane graphs. In [7], the present authors converted the Tutte polynomial of signed graphs to the chain polynomial [16] of their reductions. Now we use the above results to compute the Jones polynomial of $D^n(C')$. We first recall the definition of the chain polynomial.

Let G be a labeled graph, i.e. the graph whose edges have been labeled with elements of a commutative ring. We usually identify the edges with their labels for convenience. *The chain polynomial $Ch[G]$ of G is defined as*

$$Ch[G] = \sum_{Y \subset E} F_{G-Y}(1-w) \prod_{a \in Y} a \tag{2}$$

where the sum is over all subsets of $E = E(G)$, $F_{G-Y}(1-w)$ denotes the flow polynomial in $q = 1 - w$ of $G - Y$, the graph obtained from G by deleting the edges in Y, and $\prod_{a \in Y} a$ denotes the product of labels in Y.

Theorem 1 *Let G be a connected plane graph, E' be the edge subset of G with $|E'| = k$. Then, with the notations given in the previous sections, we have*

$$V_{D^n(C')}(t) \doteq \frac{1}{(t+1)^{c(G)}} \sum_{j=0}^{k} (-t)^{jn} \sum_{Y \subset E, |Y \cap E'| = j} F_{G-Y}(t+2+t^{-1})(-t)^{|Y-E'|}, \tag{3}$$

where $c(G)$ is the nullity of the graph G.

Proof Suppose that the edge e_i of G is labeled with a_i for $i = 1, 2, \ldots, m$. Note that $\{Y | Y \subset E\} = \bigcup_{j=0}^{k} \{Y \subset E || Y \cap E'| = j\}$. By Theorem 2 and Corollary 1 in [7] and (2), if we replace w by $-A^4 - 1 - A^{-4}$, a_i by $(-A^{-4})^n$ if $e_i \in E'$ and $-A^{-4}$ otherwise in $Ch(G)$, then

$$\begin{aligned} [D^n(C')] &= \frac{A^{(n-1)k+m}}{(-A^2 - A^{-2})^{c(G)}} Ch[G] \\ &= \frac{A^{(n-1)k+m}}{(-A^2 - A^{-2})^{c(G)}} \sum_{Y \subset E} F_{G-Y}(A^4 + 2 + A^{-4})(-A^{-4})^{|Y \cap E'|n + |Y-E'|} \\ &= \frac{A^{(n-1)k+m}}{(-A^2 - A^{-2})^{c(G)}} \\ &\quad \times \sum_{j=0}^{k} (-A^{-4})^{jn} \sum_{Y \subset E, |Y \cap E'| = j} F_{G-Y}(A^4 + 2 + A^{-4})(-A^{-4})^{|Y-E'|}. \end{aligned}$$

According to (1) and by replacing A by $t^{-1/4}$, the theorem is established. □

Remark 1 Note that in (3), for each $j = 0, 1, \ldots, k$, the summation $\sum_{Y \subset E, |Y \cap E'| = j} F_{G-Y}(t + 2 + t^{-1})(-t)^{|Y-E'|}$ is independent of n and is a Laurent polynomial in t in general.

4 Two Analytic Results

In this section we review two analytic results on zeros of polynomials, which will be used in the next two sections.

1. (Rouché's Theorem)
   Let $f(t)$ and $g(t)$ be both polynomials with complex coefficients and C be a Jordan curve in the complex plane. If $|f(t)| \leq |g(t)|$ for all $t \in C$, then the polynomials $f(t) + g(t)$ and $g(t)$ have the same number of zeros in the interior of C, counting multiplicity.
   The above result can be found in the standard textbooks on complex analysis.
2. (Beraha-Kahane-Weiss's Theorem)
   Suppose $\{f_n(x) | n = 1, 2, \ldots\}$ is a family of polynomials, a complex number z is said to be the limit of zeros of $\{f_n(x) | n = 1, 2, \ldots\}$ if either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\Re(\{f_n(x)\})$, where $\Re(\{f_n(x)\})$ is the union of the zeros of the $f_n(x)$'s.
   In [1], Beraha, Kahane and Weiss proved that if $\{f_n(x) | n = 1, 2, \ldots\}$ is a family of polynomials such that

$$f_n(x) = \alpha_1(x)\lambda_1(x)^n + \alpha_2(x)\lambda_2(x)^n + \cdots + \alpha_l(x)\lambda_l(x)^n, \tag{4}$$

where the $\alpha_i(x)$ and $\lambda_i(x)$ are fixed non-zero polynomials, such that no pair $i \neq j$ has $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some complex number ω of unit modulus. Then z is a limit of zeros of $\{f_n(x)|n = 1, 2, \ldots\}$ if and only if

(1) two or more of the $\lambda_i(z)$ are of equal modulus, and strictly greater in modulus than the others; or
(2) for some j, the modulus of $\lambda_j(z)$ is strictly greater than those of the others, and $\alpha_j(z) = 0$.

This Theorem can also be found in Sect. 3 of [5] and [3]. We call the limits of zeros in (2) of Beraha-Kahane-Weiss's Theorem the isolated limits.

5 Limits of Zeros of Jones Polynomials of $\{D^n(C')|n = 1, 2, \ldots\}$

To analyze limits of zeros of Jones polynomials of $\{D^n(C')|n = 1, 2, \ldots\}$, we need the following lemma.

Lemma 1 *Let $\{f_n(x)|n = 1, 2, \ldots\}$ and $\{V_n(x)|n = 1, 2, \ldots\}$ be two family of polynomials. Let $P(f)$ and $P(V)$ be the set of limits of zeros of $\{f_n(x)|n = 1, 2, \ldots\}$ and $\{V_n(x)|n = 1, 2, \ldots\}$, respectively. If $V_n(x) \doteq \frac{1}{(x-z_0)^l} f_n(x)$ for a fixed positive integer l, i.e. $(x - z_0)^l$ divides $f_n(x)$ for each n, and $V_n(0), f_n(0) \neq 0$ for each n, then $P(f) = P(V) \cup \{z_0\}$.*

Proof Suppose z is a limit of zeros of $\{V_n(x)\}$, then, by the definition, either $V_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\Re(\{V_n(x)\})$. If it is the first case, note that $f_n(x) \doteq (x - z_0)^l V_n(x)$, $f_n(z) = 0$ also holds for all sufficiently large n; if it is the second case, note that $\Re(\{f_n(x)\}) = \Re(\{V_n(x)\}) \cup \{z_0\}$, the two sets have the same limit points, so z is also a limit of zeros of $\{f_n(x)\}$. We have shown that if $z \in P(V)$, then $z \in P(f)$. It's clear that z_0 is a limit of zeros of $\{f_n(x)\}$. Conversely, if $z \neq z_0$ is a limit of zeros of $\{f_n(x)\}$, either $f_n(z) = 0$ for all sufficiently large n or z is a limit point of the set $\Re(\{f_n(x)\})$. For the first case, $V_n(z) \doteq \frac{1}{(z-z_0)^k} f_n(z) = 0$ also holds for all sufficiently large n; and for the second case, z is also a limit of zeros of $\{V_n(x)\}$ since $\Re(\{f_n(x)\}) = \Re(\{V_n(x)\}) \cup \{z_0\}$. □

Theorem 2 *Let G be a 2-edge connected loopless plane graph, and E' be an edge subset of G with $|E'| = k$. Then the limits of zeros of Jones polynomials of $\{D^n(C')|n = 1, 2, \ldots\}$ are*

(1) *the unit circle $|z| = 1$,*
(2) *the zeros of $t^{c(G)} \sum_{E' \subset Y \subset E} F_{G-Y}(t + 2 + t^{-1})(-t)^{|Y|-k}$ outside the circle $|z| = 1$, and*
(3) *the zeros of $t^{c(G)} \sum_{E' \cap Y = \emptyset} F_{G-Y}(t + 2 + t^{-1})(-t)^{|Y|}$ inside the circle $|z| = 1$.*

Proof Since G is bridgeless the flow polynomial $F_G(q)$ of the graph G is a polynomial of degree $c(G)$. Note that $c(H) \leq c(G)$ for every subgraph H of G, this is because the cycle space of H is a subspace of the cycle space of G and $c(H)$ and $c(G)$ are ranks of the corresponding cycle spaces, for details, see Chap. 12 of [2]. Hence, by multiplying a factor $t^{c(G)}$ (3) becomes

$$V_{D^n(C')}(t) \doteq \frac{1}{(t+1)^{c(G)}}\{\alpha_1(t)(-t)^{kn} + \alpha_2(t)(-t)^{(k-1)n} + \cdots + \alpha_k(t)(-t)^n + \alpha_{k+1}(t)\},$$

where $\alpha_i(t)$'s are all polynomials in the common sense.

Let $f_n(t) = \alpha_1(t)(-t)^{kn} + \alpha_2(t)(-t)^{(k-1)n} + \cdots + \alpha_k(t)(-t)^n + \alpha_{k+1}(t)1^n$. In order to apply Beraha-Kahane-Weiss's Theorem, we shall show that

$$\alpha_1(t) = t^{c(G)} \sum_{E' \subset Y \subset E} F_{G-Y}(t+2+t^{-1})(-t)^{|Y|-k}$$

and

$$\alpha_{k+1}(t) = t^{c(G)} \sum_{E' \cap Y = \emptyset} F_{G-Y}(t+2+t^{-1})(-t)^{|Y|}$$

are both non-zero polynomials.

Actually, for $E' \subset Y \subset E$, the degree of the term with highest degree of the polynomial $F_{G-Y}(t+2+t^{-1})(-t)^{|Y-E'|}$ is

$$\begin{aligned} c(G-Y) + |Y| - k &= (m - |Y|) - |V(G)| + k(G-Y) + |Y| - k \\ &= m - k + (k(G-Y) - |V(G)|) \\ &\le m - k, \end{aligned}$$

where equality holds if and only if $Y = E$ since G has no loops, which proves that $\alpha_1(t)$ must be a non-zero polynomial.

Similarly, for $Y \subset E$ and $Y \cap E' = \emptyset$, the degree of the term with the lowest degree of $F_{G-Y}(t+2+t^{-1})(-t)^{|Y|}$ is

$$\begin{aligned} -c(G-Y) + |Y| &= -(m - |Y|) + |V(G)| - k(G-Y) + |Y| \\ &= -m + |V(G)| + 2|Y| - k(G-Y) \\ &\ge -m + |V(G)| - 1 \\ &= -c(G), \end{aligned}$$

where equality holds if and only if $Y = \emptyset$ since G has no bridges, which proves that $\alpha_{k+1}(t)$ must be a non-zero polynomial and $\alpha_{k+1}(0) \neq 0$.

Applying (1) of Beraha-Kahane-Weiss's Theorem, we immediately get the limits $|z| = 1$. Applying (2) of Beraha-Kahane-Weiss's Theorem, isolated limits are obtained as (2) and (3) in this theorem. Note that $f_n(0) = \alpha_{k+1}(t) \neq 0$. By Lemma 1, -1 may not be a limit of zeros of Jones polynomials of $\{D^n(C')|n = 1, 2, \ldots\}$. However, since the neighboring points of -1 on $|z| = 1$ are all limits of zeros, -1 must be also a limit of zeros. This completes the proof of the theorem. □

From Theorem 2, we know that in general there are limits of zeros inside and outside the unit circle $|z| = 1$. In a particular case we refine the result and obtain

Corollary 1 *Let G be a 2-edge connected loopless plane graph. Then the limits of zeros of Jones polynomials of $\{D^n(C)\}|n = 1, 2, \ldots$ (i.e. links corresponding to graphs obtained from G by subdividing all edges uniformly) are*

(1) *the unit circle $|z| = 1$, and*
(2) *the zeros of $t^{c(G)} F_G(t+2+t^{-1})$ inside $|z| = 1$.*

Proof In this special case, $E' = E$. If $E' \subset Y \subset E$, then $Y = E$. Hence,

$$t^{c(G)} \sum_{E' \subset Y \subset E} F_{G-Y}(t+2+t^{-1})(-t)^{|Y|-k} = t^{c(G)} F_{G-E}(t+2+t^{-1})(-t)^0 = t^{c(G)},$$

which has no zeros outside $|z| = 1$. □

6 Zeros of the Jones Polynomial of $D^n(C)$

In [3], J.I. Brown and C.A. Hickman studied chromatic roots of large subdivisions of graphs. In this section, we use their method to study zeros of the Jones polynomial of $D^n(C)$ with large n. We obtain:

Theorem 3 *For any $\epsilon > 0$, there is a natural number N_ϵ such that if $n > N_\epsilon$, all zeros of Jones polynomial of the link $D^n(C)$ lie inside the circle $|z| = 1 + \epsilon$.*

Proof Note that (after omitting $\frac{1}{(t+1)^{c(G)}}$)

$$V_{D^n(C)}(t) \doteq (-t)^{mn+c(G)} + (-t)^{(m-1)n}\beta_1(t) + \cdots + \beta_{m-1}(t)(-t)^n + \beta_m(t), \tag{5}$$

where $\beta_i(t)$'s are polynomials. Let

$$f_n(t) = (-t)^{mn+c(G)}$$

and

$$g_n(t) = (-t)^{(m-1)n}\beta_1(t) + \cdots + (-t)^n\beta_{m-1}(t) + \beta_m(t).$$

We first show that all zeros of the family $\{V_{D^n(C)}(t)|n = 1, 2, \ldots\}$ are bounded.

Let $|t| = R' > 1$. Since the degree of $f_n(t)$ is $mn + c(G)$ and that of $g_n(t)$ is no more than $(m-1)n + 2c(G)$ (This is because

$$\beta_i(t) = (-t)^{c(G)} \sum_{|Y|=m-i} F_{G-Y}(t+2+t^{-1}),$$

hence its degree is no more than $2c(G)$.), there exists a natural number N such that for every $n > N$, $|f_n(t)| > |g_n(t)|$. It follows from the Rouché's Theorem that $V_{D^n(C)}(t)$ and $f_n(t)$ have the same number of zeros in the disc $|z| \le R'$, hence all zeros of $V_{D^n(C)}(t)$ lie in the disc $|z| \le R'$. Since the number of distinct zeros of all $V_{D^n(C)}(t)$'s with $n \le N$ is finite, the modulus of zeros of the family $\{V_{D^n(C)}(t)|n = 1, 2, \ldots\}$ have upper bound, say R.

For any given $\epsilon > 0$, let L be a bound for the maximum modulus of the β_i's on $1 + \epsilon \le |t| \le R$ and let t be such that $1 + \epsilon \le |t| \le R$. Then we have

$$\begin{aligned}
|V_{D^n(C)}(t)| &\ge |t|^{mn+c(G)} - |t|^{(m-1)n}|\beta_1(t)| - \cdots - |t|^n|\beta_{m-1}(t)| - |\beta_m(t)| \\
&\ge |t|^{mn} - L(|t|^{(m-1)n} + \cdots + |t|^n + 1) \\
&= |t|^{mn} - L\frac{|t|^{mn} - 1}{|t|^n - 1} \\
&= \frac{|t|^{(m+1)n} - (L+1)|t|^{mn} + L}{|t|^n - 1}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{|t|^{mn}(|t|^n - (L+1)) + L}{|t|^n - 1} \\
&\geq \frac{(1+\epsilon)^{mn}((1+\epsilon)^n - (L+1)) + L}{R^n - 1} \\
&> 0,
\end{aligned}
$$

as n is large enough. Thus there are no zeros in the annulus $1+\epsilon \leq |z| \leq R$. This completes the proof of the theorem. □

To end this section we give two remarks on Theorem 3.

Remark 2 If $R(L)$ is the mirror image of L, then a basic property of the Jones polynomial is [9, 10]

$$V_{R(L)}(t) = V_L(t^{-1}), \tag{6}$$

which tells us that zeros of Jones polynomials for L and $R(L)$ are symmetric with respect to the circle $|t| = 1$. Let $R(D^n(C))$ be the mirror image of $D^n(C)$. Then Theorem 3 becomes: For any $\epsilon > 0$, there is a natural number N_ϵ such that if $n > N_\epsilon$, all zeros of Jones polynomial of the link $R(D^n(C))$ lie outside the circle $|z| = 1 - \epsilon$.

Remark 3 Let L be an alternating link with N crossings, $\lambda_1, \lambda_2, \ldots, \lambda_N$ be zeros of $V_L(t)$ and let $\|L\| = |\lambda_1| + |\lambda_2| + \cdots + |\lambda_N|$. In [15], Lin Xiao-song posed the following conjecture: $\|L\|$ has an upper bound in the order of $N^{1+\epsilon}$. He proved the conjecture holds for many families of links in [15]. From Theorem 3, we can easily obtain that the conjecture holds for the link $D^n(C)$ if n is large enough.

7 Concluding Remarks

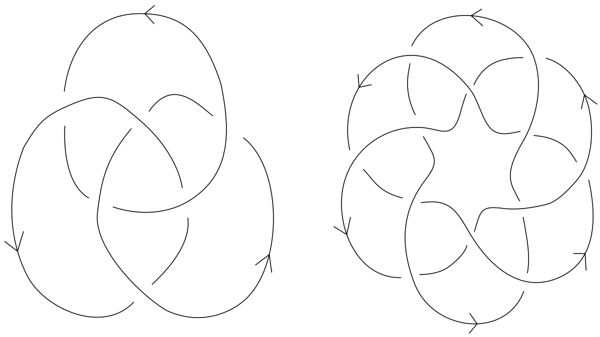
Given a connected *non-alternating* link diagram N and the subset C' of crossings of N, we can define a *non-alternating* crossing-twisted link family $\{N^n(C') | n = 1, 2, \ldots\}$ similarly. To a particular N we can use similar method in proofs of Theorems 2 and 3 to obtain similar results. The limits of zeros of Jones polynomials of $\{N^n(C') | n = 1, 2, \ldots\}$ should also be the unit circle $|z| = 1$ and some isolated points in general.

We point out that there exists non-multiple crossing-twisted link family, which also has $|z| = 1$ as its limits of zeros of their Jones polynomials. Let $m \geq 3$ be a fixed integer. Then the (m, n)-tours knots family [19] is an example. Now we provide a non-alternating link family. Let W_n be the wheel graph with $n+1$ vertices, whose spokes are labeled with signs $-$ and rim edges labeled with signs $+$. Let T_n be the link diagram corresponding to W_{3n} under the well-known one-to-one correspondence between link diagrams and signed plane graphs [13]. For example, T_1 and T_2 are shown in Fig. 4. It is not difficult to see that T_n is a 3-component link. Orienting its each component anticlockwise, we can obtain

$$V_{T_n}(t) \doteq 2t^{3n+1} + t^2 + 1. \tag{7}$$

Equation (7) can be found in [8]. By Beraha-Kahane-Weiss's Theorem, the limits of zeros of Jones polynomials of $\{T_n | n = 1, 2, \ldots\}$ are unit circle $|t| = 1$.

Fig. 4 The oriented link diagram T_n with $n = 1, 2$



Actually in general, let L_m be the link obtained from the link L by adding m full twists on n strands of L. In [4] A. Champanerkar and L. Kofman considered the Mahler measure of Jones polynomials under twisting and proved the following result (Theorem 2.5): Let $\{\gamma_i^m\}$ be the set of distinct roots of $V_{L_m}(t)$. Then, for any $\varepsilon > 0$, there is a number N_ε (independent of m) such that

$$\#\{\gamma_i^m : ||\gamma_i^m| - 1| \geq \varepsilon\} < N_\varepsilon, \tag{8}$$

as $m \to \infty$. This result is also extended to Jones polynomials of multiple twisted links: Given any link diagram L, construct $L_{n_1,\ldots,n_s}$ by adding n_i full twists on m_i strands of L for $i = 1, 2, \ldots, s$.

It is not difficult to see that the multiple crossing-twisted link $D^n(C')$ in this paper is a special case of the multiple twisted links defined above, i.e. the case $m_i = 2$ for all i. In this case our results strengthen (8) and are more particular since we proved that

1. Conversely, for any z with $|z| = 1$, it is a limit of zeros of Jones polynomials of the multiple crossing-twisted link family $\{D^n(C')|n = 1, 2, \ldots\}$.
2. For any member of the link family $\{D^n(C)|n = 1, 2, \ldots\}$, all zeros of its Jones polynomials lie within the circle $|z| = 1 + \epsilon$ if n is large enough.

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